Initial Value Problems for ODE's (IVP's)

We want numerical methods for solving ODE's of the form:

$$\left.\begin{aligned} \frac{dy}{dt} &= f(t,y) \\ \text{for } a \le t \le b \quad &\text{with} \quad y(t_0) = y_0 \end{aligned}\right\} \quad \text{———} (*)$$

i.e., we want numerical methods to approximate the solution $y(t)$ of $(*)$.

More generally:

n^{th} order ODE

$$y^{(n)} = f(t, y', y'', \ldots, y^{(n-1)})$$

for $t \in [a,b]$ with $y(a) = \alpha_1$, $y'(a) = \alpha_2$, $\ldots$ $y^{(n-1)}(a) = \alpha_n$

and

System of ODEs

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \ldots, y_n) \; ; \; i = 1, \ldots, n$$

with $t \in [a,b]$, $y_i(a) = \alpha_i$, $i = 1, \ldots, n$

One basic idea we will follow is to approximate the sol'n at certain pts $\tilde{y}(t_1), \ldots, \tilde{y}(t_m)$

and use interpolation to get the function $\tilde{y}(t) \approx y(t)$.

Before we dive in, we need some background theory:

Def'n: $f(t,y)$ is Lipschitz in y on $D \subset \mathbb{R}^2$, with constant L, if

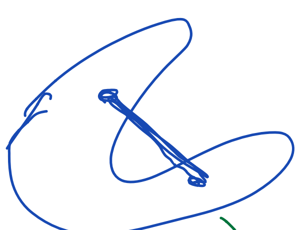
$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

$\forall (t,y_1),(t,y_2) \in D$.

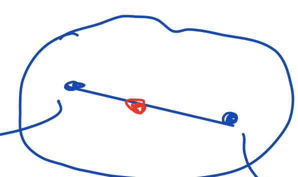
Example: $f(t,y) = t|y|$; $D = [1,2]\times[-3,4]$ is Lipschitz bec.

$$|f(t,y_1) - f(t,y_2)| = |t|\,\big||y_1| - |y_2|\big|$$
$$\leq \underbrace{2}_{\text{Lipschitz const.}}|y_1 - y_2| \quad \blacksquare$$

$\boxed{\text{Def'n: } D \subset \mathbb{R}^2 \text{ is convex if } \forall (t_1,y_1) \,\&\, (t_2,y_2) \in D \text{ we have } (1-\lambda)(t_1,y_1) + \lambda(t_2,y_2) \in D \quad \forall \lambda \in [0,1].}$



not a convex set.



(t_1,y_1) (t_2,y_2)

convex set

Theorem: If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies
convex

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L \quad \forall (t,y) \in D$$

then f is Lipschitz on D with constant L.

Sufficient cond. for a fct to be Lipschitz.

IVP Existence & Uniqueness of Sol'ns:

Theorem 1: (Existence)

If f is cont's on the rectangle

$$R = \{(t,y) : |t-t_0| \leq \alpha,\ |y-y_0| \leq \beta\}$$

then the IVP$^{(*)}$ has a sol'n $y(t)$

for $|t-t_0| \leq \min(\alpha, \beta/M)$

$M = \max_{(t,y)\in R} |f|$

__Example:__ Show that the IVP

$$\begin{cases} y' = (t+\sin y)^2 \\ y(0) = 3 \end{cases}$$

has a sol'n on $t \in [-1,1]$

__Sol'n:__ The function $f = (t+\sin y)^2$

is cont's on $R = \left\{(t,y):\ \begin{matrix} t \le \alpha \\ |y-3| \le \beta \end{matrix}\right\}$

(This is true for all $\underbrace{\alpha, \beta \ge 0}$) so we get to choose them later

and $\underbrace{\max_{(t,y)\in R} |f(t,y)|}_{M} \le (\alpha+1)^2$

so the IVP has a sol'n for

$|t| \le \min\left(\alpha, \frac{\beta}{(\alpha+1)^2}\right)$. Pick $\alpha = 1$ & $\beta = 4$

$\Rightarrow$ IVP has a sol'n for $|t| \le 1$.
(By Theorem 1)

Theorem 2 (Uniqueness):

If f and $\frac{\partial f}{\partial y}$ are cont's in

$$R = \{ (t,y) \,,\; -\alpha \le (t-t_0) \le \alpha \,, \; \beta \le (y-y_0) \le \beta \}$$

then the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

has a unique sol'n for

$$|t-t_0| \le \min(\alpha, \beta/M)$$

$$M = \max_{(t,y)\in R} |f|$$

Note that the interval may be smaller than the base of the rectangle R.

Here is a different theorem:

Theorem: Let $D = [a,b] \times \mathbb{R}$ and let f be continuous on D. If f is Lipschitz in the variable y on D then the IVP

$$\begin{cases} y'(t) = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$

has a unique sol'n $y(t)$ for $t \in [a,b]$

Example: Use the theorem to show that $\exists$ a unique solution to

$$\begin{cases} y'(t) = 1 + t\sin(ty), & 0 \le t \le 2 \\ y(0) = 0 \end{cases}$$

Sol'n: The function $f(t,y) = 1 + t\sin(ty)$ is continuous on $[0,2] \times \mathbb{R}$ & has $\left|\frac{\partial f}{\partial y}\right| = \left|t^2 \cos(ty)\right| \le 4$ on $t \in [0,2]$

$\Rightarrow$ f is Lipschitz with const 4

so by the theorem, the IVP has a unique sol'n.

Well Posedness:

The IVP : $\begin{cases} \dfrac{dy}{dt} = f(y,t), \ t\in[a,b] \\ y(a) = \alpha \end{cases}$

is well-posed if

(1) $\exists$ a unique sol'n $y(t)$

(2) The perturbed problem

$$\begin{cases} \dfrac{dz}{dt} = f(z,t) + \delta(t), \ t\in[a,b] \\ z(a) = \alpha + \delta_0 \end{cases}$$

also has a unique solution

$z(t)$ with $|z(t)-y(t)| < k\varepsilon$

$\forall$ continuous $\delta(t)$ with $|\delta(t)| < \varepsilon$

& $\delta_0 < \varepsilon$

$\in [a,b]$

Why well-posedness?

- Modeling errors
- Round-off errors
- measurement errors ...

Theorem: If f is cont. on D where $D = [a,b] \times \mathbb{R}$, and if f is Lipschitz in y on D, the IVP:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$

is well posed.

Euler's Method:

IVP:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$

- Pick $N+1$ equispaced points
  $t_0, \dots, t_N \leftarrow$ mesh points
  with
  $$t_i = a + (i-1)\underset{\uparrow \text{ step size}}{h}, \quad \text{for } h = \frac{b-a}{N}$$
  and note that
  $$\underbrace{y(t_{i+1})}_{\downarrow\; w_{i+1}} = \underbrace{y(t_i)}_{\downarrow\; w_i} + \underbrace{h\,y'(t_i)}_{\downarrow\; h\,f(t_i, w_i)} + \underbrace{\frac{h^2}{2}\,y''(\overset{\in[t_i,t_{i+1}]}{\xi_i})}_{\downarrow\; \text{approx with } 0}$$

$$\Longrightarrow \boxed{\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\,f(t_i, w_i) \\ \qquad \text{for } i = 0, \dots, N-1 \end{cases}}$$

$\uparrow$ Difference Eq'n.

Now, we have approx. values of y at $t_0, \dots, t_N$, so we can interpolate to find an approx. of y.

Example: $\begin{cases} y' = y - t^2 + 1 & t \in [0,2] \\ y(0) = 0.5 \end{cases}$

Pick $h = 1$ & use Euler's method:

$\Rightarrow t_0 = 0, \quad t_1 = 1, \quad t_2 = 2$

$w_0 = 0.5$

$$w_1 = w_0 + h f(t_0, w_0)$$
$$= 0.5 + 1 \times (0.5 - 0^2 + 1) = 2$$

$$w_2 = w_1 + h f(t_1, w_1)$$
$$= 2 + 1 \times (2 - 1^2 + 1) = 4$$

Error Bounds

Theorem: If f is Lipschitz continuous on $D = [a,b] \times \mathbb{R}$ with const. L and if

$$|y''(t)| \leq M \quad \forall t \in [a,b]$$

where y is the unique sol'n to

$\begin{cases} y' = f(t,y) & t \in [a,b] \\ y(a) = \alpha \end{cases}$

then $$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right] \quad i = 0, \ldots, N$$

Higher order Taylor methods:

Local Truncation error:

In Euler's method, we computed

$$w_{i+1} = w_i + h f(t_i, w_i)$$

More generally, we may use an iteration of the form

$$w_{i+1} = w_i + h \phi(t_i, w_i)$$

for some $f \neq \phi$ we choose.

The local truncation error can be defined as

$$\tau_{i+1}(h) := \underbrace{y_{i+1}}_{y(t_{i+1}) \text{ true value at } t_{i+1}} - \overbrace{\left(y_i + h\phi(t_i, w_i)\right)}^{\text{predicted value at } t_{i+1} \text{ starting from } y_i}$$

<u>Example</u>: Euler's method has

$$\tau_{i+1}(h) = y_{i+1} - y_i - h\underbrace{f(t_i, y_i)}_{y'(t_i)}$$

$$= \frac{h^2}{2} y''(\xi_i) \qquad \xi_i \in [t_i, t_{i+1}]$$

$$= O(h^2)$$

~x~

This suggests improving upon Euler's method by using higher order Taylor approximations:

$$(*) \quad y(t_{i+1}) = y(t_i) + \sum_{i=1}^{n} \frac{h^i}{i!} y^{(i)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

$$(**) \quad \boxed{\begin{aligned} y'(t) &= f(t, y(t)) \\ y''(t) &= f'(t, y(t)) \\ &\vdots \\ y^{(k)}(t) &= f^{(k-1)}(t, y(t)) \end{aligned}}$$

(*) & (**) $\Longrightarrow$

$$y(t_{i+1}) = y(t_i) + \sum_{i=1}^{n} \frac{h^i}{i!} f^{(i-1)}(t_i, y(t_i))$$

$$+ \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

If we ignore the error term (to get our numerical method), we have

Taylor methods of order n:

$$\boxed{\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\, T^{(n)}(t_i, w_i) \end{cases}}$$

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

(Euler = Taylor of order 1)

Example: Derive Taylor's method of order 2 with $h=1$ for the IVP

$$\begin{cases} y' = y - t^2 + 1 & t \in [0,2] \\ y(0) = 0.5 \end{cases}$$

Sol'n: we have $f(t,y) = y - t^2 + 1$ and we need to compute

$$f'(t,y) = \frac{d}{dt}\left(y(t) - t^2 + 1\right)$$

always remember that y is a fn of t!

$$= \underbrace{y'(t)}_{f(t,y)} - 2t$$

$$= y - t^2 + 1 - 2t$$

So

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$

$$= w_i - t_i^2 + 1 + \frac{h}{2}\left(w_i^2 - t_i^2 - 2t_i + 1\right)$$

$$= \left(1 + \frac{h}{2}\right)\left(w_i - t_i^2 + 1\right) - h t_i$$

$$\Rightarrow \begin{cases} w_0 = 0.5 \\ w_{i+1} = w_i + h\underbrace{\left[\left(1+\frac{h}{2}\right)\left(w_i - t_i^2 + 1\right) - h t_i\right]}_{T^{(2)}(t_i, w_i)} \end{cases}$$

Theorem: Taylor's method of order n has local truncation error $O(h^{n+1})$ provided the sol'n $y \in C^n[a,b]$.

Advantages: • high order local truncation error

Disadvantage: Must compute & evaluate derivatives of $f(t,y)$